Pricing Credit from Equity Options

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Theoretical arguments and anecdotal evidence suggest a strong link between credit quality of a particular name and its equity-related characteristics, such as share price and implied volatility. In this paper we develop a novel approach to pricing and hedging credit derivatives with equity options and demonstrate the predictive power of the model.

1 Introduction

Exploring the link between debt and equity asset classes has been and still is a subject of considerable interest among researchers and practitioners. The largest interest had been among market participants that dealt in cross asset products like convertible bonds and hybrid exotic options but also among those that are trading equity options and credit derivatives. Easy access to structural models (CreditGrades, KMV), their intuitive appeal and simple calibration enabled a historical analysis of credit market implied by equity parameterisation. Arbitrageurs are setting up trading strategies in an attempt to exploit eventual large discrepancies in credit risk assessment between two asset classes. In what follows we will present a new market model for pricing binary CDS using implied volatility extracted from equity option market instruments. We will show that the arbitrage relationship derived in this paper can be enforced via dynamic hedging with Gamma-flat risk reversals. In this framework the recovery rate is an external uncertain parameter which cannot be extracted from equity-related instrument and is defined by the debt structure of a particular company. In trading the arbitrage-related strategies, arbitrageurs have to have a view on the recovery or use analysts estimates to form their own conservative assumptions.

Existence of simple structural models made it possible for the arbitrage community to participate in the attempt to align the credit market with equity parameterised models. In the course of implementing trading strategies based on structural models it became apparent that the models are not true no-arbitrage models in the sense that they do not identify riskless arbitrage portfolios which can be traded when market CDS prices deviate from the model predictions. From this perspective, the structural models define some sort of "fair value" in the same way as Capital Asset Pricing Model defines an equilibrium "fair value" for the share returns. The step which we are taking in this paper is a derivation of a *no-arbitrage approach to valuation of credit derivatives based on existence of an arbitrage portfolio of stocks and equity options such that trading the portfolio can enforce the no-arbitrage relationship.* The reader can look at the suggested here analysis as a generalisation of the standard Black-Scholes analysis to the case of defaultable underlying assets. In this case CDS plays a role of

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a market asset which completes the market and allows to eliminate the default risk in the riskless portfolio.

In the next section we briefly describe the volatility-linked model for valuing binary cds. Technical details of the derivation are summarized in the Appendix. The model is then tested by building an arbitrage trading strategy based on the arbitrage portfolio and the construction is back-tested for a large number of actively traded credit names. The paper is concluded with remarks on possible applications of the approach to trading long-term volatility products, hedging credit derivatives with equity options and modelling forward starting equity options.

2 Credit Risk Reversal

The main idea of the model is to build an enforceable arbitrage trading strategy to hedge binary CDS with equity and equity options. The strategy has to generate an arbitrage return if the price of the binary CDS is not in line with the cost of the hedging strategy, thus prompting arbitrageurs to step in and eliminate the mispricing. We saw before that the main problem of the structural models was absence of the enforceable hedging strategy. This essentially means the absence of the price-regulating mechanism and potentially large credit-equity decouplings. In the case of an enforceable arbitrage strategy arbitrageurs enter the arbitrage position when the price deviations are large enough to cover the model risk and transaction costs with the goal to benefit from either positive carry on the position till the CDS expiry (if the deviations do not disappear) or capital gain (in case of "washing out" of the arbitrage).

The model makes the following assumptions. First, a credit event triggering the binary payment is accompanied by a single jump of share price to zero after which the trading stops. Second, before the credit event stocks and options can be traded continuously for a reasonable range of strikes. Below the range of strikes will be within Δ 15 put and Δ 15 calls range[1]. Third, in normal market conditions the stock price moves can be approximated by continuous diffusion-like process, possibly with stochastic volatility and other complexities. We also assume no transaction costs for which can be accounted later in the final result.

Let us consider a portfolio which is short a binary CDS of T years till maturity and notional equal to \$1. This instrument triggers no payment (no cash outflows) if there was no credit event and makes a single payment of \$1 in case of the event. By the definition the credit event is a dicrete event which, as follows from the assumptions above, is accompanied by a jump in stock price to zero. Therefore, to hedge the binary CDS a hedger has to build a portfolio which produces no cash flows during normal trading and produces a single payment of \$1 in case of the stock price jump to zero. The cost of carry of this hedging portfolio should be equal to the cost of carry of the binary CDS thus defining its "fair" price.

Illustrative example: case of zero implied volatility skew. The first note we would like to make relates to the connection between equity option skew and the binary CDS price. Simple analysis shows that the CDS spread should be linear in implied volatility skew and has to be zero if the skew is zero. Indeed, lets consider the following hedging strategy in the case of zero implied skew: every day we establish a position in zero-Gamma zero-Vega risk reversals (long puts and short calls) with

[1] Δ 15 put is defined as the put with strike such that that the delta on the put is equal to -15%. Similarily, the Δ 15 call a call with strike such that that the delta on the call is equal to 15%. The strikes obviously depend on the levels of the implied volatility and the tenor of the options.

symmetrical strikes K_c and K_p and maturity T:

$$K_c K_p = F_T^2$$

so that

$$d_{1,p} = -d_{2,c} \quad , \quad d_{2,p} = -d_{1,c} \; ,$$

where F_T is the corresponding forward price and $d_{1,p(c)}, d_{2,p(c)}$ are the standard Black-Scholes notations:

$$d_{1,c} = \frac{ln\,(F_T/K_c) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad , \quad d_{2,c} = d_{1,c} - \sigma\sqrt{T-t}$$

$$d_{1,p} = \frac{ln\,(F_T/K_p) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad , \quad d_{2,p} = d_{1,p} - \sigma\sqrt{T-t} \; .$$

Zero-Gamma zero-Vega condition requires the ratio of the put/call notionals to be equal to $\sqrt{K_c/K_p} = K_c/F_T$. It is not difficult to check that such risk reversal has zero initial cost to enter. If the risk reversal is delta-hedged, the only component of Daily P&L will be third moment, the realised skew. In average the realised skew is close to zero to be consistent with Edgworth expansion and our assumption of zero implied skew. In fact, even in the presence of implied volatility skew the realised skew is quite small and can be even positive. Table 1 shows historical realised skew for 10 liquid stocks in the US and Europe. This means that on average one can roll the delta-hedged risk reversal to new symmetrical puts and calls at no cost. At the same time, the dynamical risk reversal position provides the credit protection because in case of price jump to zero each put will generate $\frac{K_c K_p}{F_T} = F_T$ units of cash which after compensating loss on the long delta from the hedge, can be used to cover the binary CDS. This is where the non-linear nature of equity options step in because the convexity of the put payoff will ensure larger gain on the put than the loss on the delta hedge. Since carrying the hedging portfolio costs the hedger zero, the CDS spread has to be zero from no-arbitrage arguments. It is important to emphasize that in this case the arbitrage is actually enforceable because we establish a pricing relationship between directly tradeable instruments. This shows that the CDS spread comes from the implied volatility skew and, generally, has to compensate the carry cost of the risk reversal position which is linear in skew for sufficiently small skews. Hence the CDS price has to be linear in skew as well.

General case: non-zero implied volatility skew. To hedge binary CDS in the case of presence of implied volatility skew we will again establish a daily position in a delta-neutral risk reversal which has zero-Gamma (but non-zero Vega) and which generalizes the symmetrical case considered above. Below we call this risk reversal structure a *Credit Risk Reversal*. We introduce calls with strikes K_c with $d_{1,c} = d$ and puts with stikes K_p such that $d_{1,p} = -d_{2,c}$. The portfolio will contain $\frac{K_c}{F_T}$ long puts and $\frac{\sigma_{c,T}}{\sigma_{p,T}}$ short calls with maturity T. It is checked in the Appendix that the portfolio has zero Gamma.

To make the consideration more analytically tractable we make several approximations. We assume Log-Linear implied volatility skew:

$$\sigma_{c,T} = \sigma_{A,T} + \beta_T ln\frac{F_T}{K_c} \quad , \quad \sigma_{p,T} = \sigma_{A,T} + \beta_T ln\frac{F_T}{K_p} \; ,$$

where $\sigma_{A,T}$ is the ATM (forward) volatility for maturity T and β_T is the corresponding skew. We also use the small volatility expansion:

$$|d_{1,c}| >> \frac{1}{2}\sigma_{c,T}\sqrt{T-t} \; , |d_{1,p}| >> \frac{1}{2}\sigma_{p,T}\sqrt{T-t} \; ,$$

so that $d_{1,c} \sim d_{2,c}$ and[2]

$$ln\frac{F_T}{K_c} = d_{1,c}\sigma_{c,T}\sqrt{T-t} \quad , \quad ln\frac{F_T}{K_p} = -d_{1,c}\sigma_{p,T}\sqrt{T-t} \; .$$

The last approximation is the small skew expansion:

$$\beta|d_{1,c}|\sqrt{T-t} << 1$$

so that[3]

$$\sigma_{c,T} = \sigma_{A,T} + \beta_T d_{1,c}\sigma_{A,T}\sqrt{T-t} \quad , \quad \sigma_{p,T} = \sigma_{A,T} - \beta_T d_{1,c}\sigma_{A,T}\sqrt{T-t} \; .$$

The approximations limit the range for $|d_{1,c}|$ to $[0.6, 1.5]$. In what follows we use $|d_{1,c}| \sim 1$.

In these approximations the daily P&L on the Credit Risk Reversal which generates \$1 in case of default

$$HedgeP\&L_{daily} = \frac{dN'(d)}{Df - 2N(d)}\left(\frac{\left(\frac{\Delta S}{S}\right)^3}{3\sigma_{A,T}^2(T-t)} + 2\beta_T\sigma_{A,T} - 2(T-t)\left(d(\beta_T\sigma_{A,T}) + \beta d(\sigma_{A,T})\right)\right) \; .$$

as derived in the Appendix. If one neglects the discount factors, after integration over the life time of binary cds this gives the amount which in total has to be paid to the CDS seller for providing the credit protection (Eqn (8 in Appendix):

$$HedgeP\&L = \frac{dN'(d)}{1-2N(d)}\left(\sum_{i=t}^{T-1}\frac{\left(\frac{\Delta S_{i+1}}{S_i}\right)^3}{3\sigma_{A,T}^2(\tau_i)(T-\tau_i)} + 2(T-t)\beta_T(t)\sigma_{A,T}(t) - 2\int_t^T (T-\tau)\beta_T(\tau)d(\sigma_{A,T}(\tau))\right) \; .$$

For $d = -1$ the numerical factor in front is equal to 0.35 which corresponds to Δ 16 calls.[4]

Current price of the binary CDS is defined by expectation of the right hand side of the last equation. More precisely, the upfront price of the cds is equal to the expected cost of carry of the Credit Risk Reversal position:

$$cds_{upfront} = \frac{-dN'(d)}{1-2N(d)}\left(E_t\left[\sum_{i=t}^{T-1}\frac{\left(\frac{\Delta S_{i+1}}{S_i}\right)^3}{3\sigma_{A,T}(\tau_i)^2(T-\tau_i)}\right] + 2(T-t)\beta_T(t)\sigma_{A,T}(t) - 2\int_t^T (T-\tau)E_t(\beta_T(\tau)d(\sigma_{A,T}(\tau)))\right) \; .$$

There are three terms contributing to the price of the cds. The first one is the expected realised skew accumulated by the risk reversal position over the life of the hedging strategy. We saw above (see Table 1) that this term is on average negligable comparing with the cost of carry of the position and

[2] Example: $\sigma \sim 0.5$, $T - t = 5$ gives $\frac{1}{2}\sigma\sqrt{T-t} \sim 0.6$. $d_{1,c} = -0.67$ corresponds to $\Delta_{call} = 0.25$. This puts downside boundary on $d_{1,c}$ to be used.

[3] Example: $\beta \sim 0.3$, $T - t = 5$ gives $\frac{1}{\beta\sqrt{T-t}} \sim 1.5$. This puts upside boundary on $d_{1,c}$ to be used.

[4] Strictly speaking, T in (8) has to be equal to the minimum between the CDS maturity and expected time of default. Since the expected time to default is a function of the default probability and, therefore, the binary CDS price, this introduces non-linearity into the equation decreasing the expected hedging costs. For long-term contracts or bad credit this might be quite significant. It is possible to estimate the expected time to default as $1/p$ where p is the binary CDS price (paid annually). This time is usually much larger than the life of the contract. In what follows we ignore the complication.

we can safely ignore it. The second term is directly expresses the cost of carry as implied by the current equity option market. This term is proportional to the skew as expected and generates Vega of the CDS with respect to the equity volatility. Since $\beta_T(t)$ is approximately equal to the 1Y skew scaled by the square root of time to maturity, $\beta_{1Y}/\sqrt{T-t}$, the vega of the CDS is proportional to $\sqrt{T-t}$ in line with time dependence of vanilla options vegas.

The third term has a more complicated nature. It is generated by the expected changes in the implied volatility levels and, in principle, has to be priced from a stochastic implied volatility model. However, to get a simplified picture, we make the following approximations:

1. As above, $\beta_T(\tau) = \beta_{1Y}/\sqrt{T-\tau}$.
2. $d(\sigma_{A,T}(\tau)) = d(\sigma_{A,1Y}(\tau))/\sqrt{T-\tau}$, flat Power Vega P&L.
3. Forward skew is propagated at 1, which means that $E_t(\beta_{1Y}(\tau)) = \beta_{1Y}(t)$ and is independent on the ATM volatility level.

All these approximations are empirically sufficiently accurate to estimate the interesting term without going into much details about choice of the stochastic implied volatility model. This results in the following form of the third term

$$\int_t^T (T-\tau)E_t\left[\beta_T(\tau)d(\sigma_{A,T}(\tau))\right] = \beta_{1Y}(t)E_t(\sigma_{A,1Y}(T)) - \beta_{1Y}(t)\sigma_{A,1Y}(t) \ ,$$

which can be estimated from the current term structure. This reduces the above expression to the simple analytical form:

$$cds_{upfront} = -2\frac{dN'(d)}{1-2N(d)}\left((T-t)\beta_T\sigma_{A,T} - 2\beta_{1Y}(t)E_t(\sigma_{A,1Y}(T)) + 2\beta_{1Y}(t)\sigma_{A,1Y}(t)\right) \ .$$

This form is applied below to test predictive power of the model.

3 Testing the Model

The model described in the previous section not only identifies the "fair" value for the CDS but also prescribes the set of actions to be taken if the market value deviates from the model "fair" value thus providing means to enforce the model arbitrage. Therefore one of possible ways to test the model is to back-test an arbitrage trading strategy which is based on the model.

To examine the predictive power of the model we used historical data for CDS, stock prices and implied volatility surfaces taken from a JPMorgan proprietary database. First, the model is run to get the history of model spreads implied by option prices. Then trading signals are identified. This is done automatically by specifying the difference between theoretical and market spreads at which to enter the trade and one at which to close the position. During the life of the trade positions are delta-hedged and rolled when necessary. Sum daily P&Ls represents the cumulative P&L of the strategy for a particular name.

To explain the testing procedure we run the model for Altria Group Inc(Bloomberg: MO US). Figure 1 shows senior 5Y CDS levels against the volatility-based model at recovery rate 0.5. One can see that market CDS levels are tightly correlated with the theoretical levels dictated by the model and the basis, the difference between the market levels and the theoretical levels, is strongly mean-reverting. This allows us construct a strategy which trades the basis: buy CDS and sell the Credit Risk Reversal if the basis is negative and sell CDS and buy Credit Risk Reversal if the basis is positive. Example of the trade would be as follows.

Example. On April 4 an arbitrageur sells 5Y CDS at 545 bps for \$6 mm notional, sells 320 000 calls with strike 40 and maturity Jan 2005 (bid at 2.2, 30% delta) and buys 560 000 puts with the same maturity and the strike 15 (offered at 2.1, -16% delta). The position is Gamma-flat and is Δ-hedged by buying 185600 synthetic forwards at 24.14 level. The Break Even Recovery is 35% assuming stock goes to 0 on default. On April 15 the arbitrageur closes the position with CDS traded at 350 bps, puts traded at 1.07, calls traded at 3 and stock comming up to 32.80 and forward being at 29.47. The trade P&L is \$671250 which is about \$200000 better than P&L for the delta-hedging (E2C) strategy.

It is possible to back-test the arbitrage trading based on the Credit Risk Reversal similar to the example above. Figure 2 demonstrate running cumulative P&L on the strategy based on two year trading history. In the simulation we assumed 120 bps basis spread as an entry signal for the trade and 10 bps as the signal to close the position. The figure shows that the strategy generates positive return with relatively small P&L fluctuations. Figures 3 and 4 show the results of the analysis for basis trading for Suez SA (Bloomberg: SZE FP). The charts also demonstrate profitability of the arbitrage trading based on the Credit Risk Reversal model. Figures 5 and 6 show the results for TYCO International ltd (Bloomberg: TYC US) as an example of running the strategy for a cross-over credit name.

We have run the analysis for 20 actively traded credit names in Europe and 20 names in the US. All them demonstarte qualitatively similar results. Overall the strategy is profitable and generates relatively small P&L noise. This demonstrates that the arbitrage model for CDS pricing in terms of dynamical equity option strategy correctly identifies the fair value of CDS and is suitable for enforcing the no-arbitrage relationship as it is claimed in the technical part of the paper. We also compared trading performance of the Credit-Volatility strategy with performances of standard debt-equity trading strategies. Table 2 shows three P&Ls on trades in Suez SA which are motivated by CeditGrades-based strategy, static CDS vs puts strategy and the Credit-Volatility strategy. The Table shows that the latter produces superior result comparing with other strategies.

4 Discussion of Possible Applications

The Credit-Volatility model presented in this paper allows one to identify profitable mispricing opportunities between credit and equity option markets and, what is more important, prescribes a course of actions to extract the arbitrage return in the case of the mispricing. These actions do not depend on whether the mispricing persists or not and it is not particularly sensitive to the choice of the model. We saw above that the US market shows tighter relationship between the equity options and CDS, with shorter mean-reversion times and a smaller decoupling. Names in Europe demonstrate less tight picture thus providing bigger potential for relative value trades. As more market participants recognize the relationship, it becomes more tight and stable which will diminish arbitrage returns but opens new interesting applications of the credit-volatility connection. We list here three possible directions which appear to be especially promising.

1. Long-term volatility products. It follows from our analysis that credit derivatives are a source of long term implied volatility. It must be said that most of equity options are liquid up to 2-3 years maturity and it is not immediately clear how long-term options have to be priced and hedged.

In this respect, credit derivatives, which are most liquid in 5 years but are available for much longer maturities, can be used effectively to hedge options vega exposure if some conservative assumptions about recovery rates are made.

2. Hedging credit derivatives with equity options. Currently most models of options on credit default swaps are valued in the Black-Scholes framework which implies a possibility of continuous hedging. At the same time, the underlying CDS are traded in relatively large size, typically 3-5 mln dollars. This precludes the continuous hedging and leaves swaptions market makers with considerable market risk. Equity options can be traded in small size thus allowing quasi-continuous dynamical hedging of swaptions, if the credit-relationship is sufficiently tight.

3. Forward starting equity options. Trading credit default swaps for different maturities it is possible to create forward starting CDS position. Applying the credit-volatility model to the pricing of forward starting CDS one can establish boundaries for forward starting implied volatilties which can be used to price and hedge forward starting options. In this way the term structure of credit default swaps sheds light on structure of forward ATM volatilities and forward skews.

5 Appendix. Cost of Carry for Credit Risk Reversal

In this Appendix we derive the expression for the cost of carry of Credit Risk Reversal used in the main text. As explained above, to hedge binary CDS in presence of implied volatility skew one has to establish a daily position in delta-neutral risk reversal with zero Gamma but generally non-zero Vega. The new risk reversal consists of $\frac{\sigma_c}{\sigma_p}$ short position in calls with strikes K_c with $d_{1,c} = d$ and $\frac{K_c}{F}$ long position in puts with stikes K_p such that $d_{1,p} = -d_{2,c}$.

First of all, it is easy to check that the portfolio has zero Gamma. Indeed,

$$S^2\Gamma_{call} = Df^{-1}F_T\frac{N'(d_{1,c})}{\sigma_{c,T}\sqrt{T-t}}$$

and

$$S^2\Gamma_{put} = Df^{-1}F_T\frac{N'(d_{1,p})}{\sigma_{p,T}\sqrt{T-t}}$$

where Df is the corresponding discount factor. Since $d_{1,p} = -d_{2,c}$ one can rewrite the last relation as

$$S^2\Gamma_{put} = Df^{-1}F_T\frac{N'(d_{1,c})}{\sigma_{p,T}\sqrt{T-t}}e^{\sigma_{c,T}\sqrt{T-t}d_{1,c}-\frac{1}{2}\sigma^2_{c,T}(T-t)} \ .$$

Using the explicit form for $d_{1,c}$ we arrive at the expression

$$S^2\Gamma_{put} = Df^{-1}F_T\frac{N'(d_{1,c})}{\sigma_{c,T}\sqrt{T-t}}e^{-ln(K_c/F_T)}\frac{\sigma_{c,T}}{\sigma_{p,T}} \ .$$

which provides that

$$\frac{K_c}{F_T}\Gamma_{put} - \frac{\sigma_{c,T}}{\sigma_{p,T}}\Gamma_{call} = 0 \ .$$

Therefore the Gamma of the portfolio is equal to zero.

The next step is to calculate Vega or, rather, daily Vega P&L of the portfolio. This is easy to do if one uses the relationship

$$Vega_{call} = S^2\Gamma_{call}\sigma_{c,T}(T-t) \ ,$$
$$Vega_{put} = S^2\Gamma_{put}\sigma_{p,T}(T-t) \ .$$

The relationship gives the following Vega P&L for the portfolio

$$VegaP\&L = Df^{-1}F_TN'(d_{1,c})\sqrt{T-t}\left(d(\sigma_{p,T}-\sigma_{c,T}) + d(\sigma_{c,T})(1-\frac{\sigma_{c,T}}{\sigma_{p,T}})\right) \ . \tag{1}$$

Theta for the portfolio is a little more complicated.

$$\theta_{call} = -Df^{-1}F_T\frac{\sigma_{c,T}}{2\sqrt{T-t}}N'(d_{1,c}) + DivDf^{-1}F_TN(d_{1,c}) - rK_cDfN(d_{2,c}) \ ,$$

$$\theta_{put} = -Df^{-1}F_T\frac{\sigma_{p,T}}{2\sqrt{T-t}}N'(-d_{1,p}) - DivDf^{-1}F_TN(-d_{1,p}) + rK_pDfN(-d_{2,p}) \ .$$

Our portfolio can be delta-hedged with forwards so that one can ignore the dividend component of Theta. We also neglect the interest rate component which would result in the following expression for the Theta of the portfolio:

$$\theta_{portfolio} = Df^{-1} F \frac{N'(d_{1,c})}{2\sqrt{T-t}} \left(\frac{\sigma_{c,T}^2}{\sigma_{p,T}} - \sigma_{p,T} \right) . \tag{2}$$

The last bit we will need is the delta of the Credit Risk Reversal:

$$S\Delta_{CRR} = Df^{-1} F_T \left(\frac{K_c}{F_T} (N(-d_{2,c}) - 1) - \frac{\sigma_{c,T}}{\sigma_{p,T}} N(d_{1,c}) \right) . \tag{3}$$

Under the three approximations described in the main text, the P&L components will be equal to:

$$VegaP\&L = Df^{-1} F_T N'(d_{1,c}) \sqrt{T-t} \left(-2\sqrt{T-t} \cdot d_{1,c} \cdot d(\beta_T \sigma_{A,T}) - 2\sqrt{T-t} d_{1,c} \left(\frac{\beta_T \sigma_{A,T}}{\sigma_{A,T}} \right) d(\sigma_{A,T}) \right) , \tag{4}$$

$$\theta_{portfolio} = Df^{-1} F_T \frac{N'(d_{1,c})}{2\sqrt{T-t}} \left(\frac{(\sigma_{c,T} - \sigma_{p,T})(\sigma_{c,T} + \sigma_{p,T})}{\sigma_{p,T}} \right) = 2Df^{-1} F_T N'(d_{1,c}) d_{1,c} (\beta_T \sigma_{A,T}) . \tag{5}$$

To address the notional of the portfolio one has to consider the default scenario. In case of default the delta-hedged portfolio has to generate \$ 1 to cover the payment on the binary CDS. Therefore the notional has to be

$$\frac{1}{F_T \frac{K_c K_p}{F_T^2} - (1 + e^{-d_{1,c}\sigma_{c,T}\sqrt{T-t}}) F_T Df^{-1} N(d_{1,c})}$$

which in low order on ATM volatility and skew order [5] gives

$$\frac{Df}{F_T} \frac{1}{Df - 2N(d_{1,c})} . \tag{6}$$

Let's return back to the Gamma P&L. We showed earlier that the Gamma of the portfolio is equal to zero. However, this does not imply that daily Gamma P&L will be equal to zero because the price distribution can have a non-zero third moment which will contribute to the P&L.

Suppose we rehedge daily and Γ on the position is equal to zero at last close. This implies that

$$\Gamma = -\alpha(S - S_0)$$

and

$$\Gamma P\&L = -\frac{\alpha}{6}(S - S_0)^3 .$$

Since

$$\frac{\partial \Gamma_{call}}{\partial S} = -\frac{F_T Df^{-1}}{S^3} \frac{N'(d_{1,c})}{\sigma_{c,T}\sqrt{T-t}} - \frac{F_T Df^{-1}}{S^3} \frac{d_{1,c} N'(d_{1,c})}{\sigma_{c,T}^2 (T-t)}$$

[5] The P&L components are already proportional to the skew and ATM volatility. Thus in the small skew/small vol expansion the skew effect on the notional can be ignored. We also neglect cost of establishing CRR position because it is too proportional to the skew.

and

$$\frac{\partial\Gamma_{put}}{\partial S} = -\frac{F_T Df^{-1}}{S^3}\frac{N'(d_{1,p})}{\sigma_{p,T}\sqrt{T-t}} - \frac{F_T Df^{-1}}{S^3}\frac{d_{1,p}N'(d_{1,p})}{\sigma_{p,T}^2(T-t)} \ ,$$

one obtains

$$\frac{\partial\Gamma_{portfolio}}{\partial S} = \frac{F_T Df^{-1}}{S^3}\frac{d_{1,c}N'(d_{1,c})}{(T-t)}\left(\frac{1}{\sigma_{p,T}^2} + \frac{1}{\sigma_{p,T}\sigma_{c,T}}\right) \ .$$

Therefore in our case the constant α is equal to

$$\alpha = -2\frac{F_T Df^{-1}}{S^3}\frac{d_{1,c}N'(d_{1,c})}{\sigma_{A,T}^2(T-t)}$$

and the daily Gamma P&L has the form:

$$\Gamma P\&L = 2F_T Df^{-1}\frac{d_{1,c}N'(d_{1,c})}{6\sigma_{A,T}^2(T-t)}\left(\frac{\Delta S}{S}\right)^3 \tag{7}$$

Collecting together equations (4), (5), (7) and (6) we obtain the following daily cost of hedging of our binary CDS:

$$HedgeP\&L_{daily} = \frac{d_{1,c}N'(d_{1,c})}{Df - 2N(d_{1,c})}\left(\frac{\left(\frac{\Delta S}{S}\right)^3}{3\sigma_{A,T}^2(T-t)} + 2\beta_T\sigma_{A,T} - 2(T-t)\left(d(\beta_T\sigma_{A,T}) + \beta_T d(\sigma_{A,T})\right)\right) \ .$$

Integrating over time one can find the total cost of carry for the hedging position and, therefore, the total payment for the CDS protection. This means that, ignoring discount factors, the CDS price (paid upfront) is equal to the expected cost of hedging

$$cds_{upfront} = \frac{-d_{1,c}N'(d_{1,c})}{1-2N(d_{1,c})}E_t\left(\sum_{i=t}^{T-1}\frac{\left(\frac{\Delta S_{i+1}}{S_i}\right)^3}{3\sigma_{A,T}^2(\tau_i)(T-\tau_i)} - 2\int_t^T d((T-\tau)\beta_T(\tau)\sigma_{A,T}(\tau_i)) - 2\int_t^T (T-\tau)\beta_T(\tau)d(\sigma_{A,T}(\tau))\right) \ ,$$

or, after integration,

$$cds_{upfront} = \frac{-d_{1,c}N'(d_{1,c})}{1-2N(d_{1,c})}E_t\left(\sum_{i=t}^{T-1}\frac{\left(\frac{\Delta S_{i+1}}{S_i}\right)^3}{3\sigma_{A,T}^2(\tau_i)(T-\tau_i)} + 2(T-t)\beta_T(t)\sigma_{A,T}(t) - 2\int_t^T (T-\tau)\beta_T(\tau)d(\sigma_{A,T}(\tau))\right) \ . \tag{8}$$

Figures

Figure 1. Market levels of 5Y cds on Altria Group Inc versus theoretical (volatility-based) value. Recovery value taken is 0.5.

Figure 2. Cumulative historical P&L of the arbitrage basis trading strategy for Altria Group Inc. Entry difference is 120, exit difference is 10.

Figure 3. Market levels of 5Y cds on Suez SA versus theoretical (volatility-based) value. Recovery value taken is 0.75.

Figure 4. Cumulative historical P&L of the arbitrage basis trading strategy for Suez SA. Entry difference is 50, exit difference is 5.

Figure 5. Market levels of 5Y cds on TYCO International Ltd versus theoretical (volatility-based) value. Recovery value taken is 0.40.

Figure 6. Cumulative historical P&L of the arbitrage basis trading strategy for TYCO International Ltd. Entry difference is 110, exit difference is 10.

Tables

Table 1. Normalized third moment $< \left(\frac{dS}{S}\right)^3 > / < \left(\frac{dS}{S}\right)^2 >$ of historical daily returns for some active credit names in Europe and the US. The table gives 6 months and 1 year historical third moments averaged over last four years. The reader can see that the contribution from the moment to the CDS price is small and on average can be neglected.

Name	6*M*	1*Y*
MOUS	0.004	0.011
FUS	0.011	0.006
GMUS	−0.009	−0.003
IBMUS	−0.003	−0.001
JPMUS	−0.013	−0.012
DTEGR	0.017	0.012
SIEGR	0.011	0.009
FTEFP	0.082	0.071
EXFP	−0.055	−0.055
UBSNVX	0.003	0.003

Table 2. Table shows results of debt-equity trades for Suez SA based on CeditGrades (1), static CDS vs puts strategy (2) and the Credit-Volatility strategy(3). The latter strategy produces superior result comparing with first two strategies.