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Lies of Capital Lines

Kirill Ilinski [1], Alexis Pokrovski [2]

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Absract

In this paper we examine in detail the qualitative effects caused by the investors sensitivity to mark-to-market and price of liquidity. This distorts CAPM-like portfolio construction causing the Capital Line to become curved and eventually inverted. We show that in the world of strongly concave and non-monotonous Capital Lines, pushing up return targets results in increasing risk but not in increasing return. This is due to the decreasing and eventually negative marginal returns per unit of risk. By chasing returns and prompting investment managers to deliver unsustainable performance, the investment community damages its own chances through a greedy search for yield. Besides negatively skewing the risk-return characteristics, this also amplifies destabilizing pro-cyclical dynamics and increases the component of volatility, which is not accompanied by corresponding return. The latter has profound consequences for investment management and economic policy making. We examine the influence of non-linearity of the Capital Line on the cyclical volatility of capital market and short optionality ("negative gamma") profile, implicitly embedded in classical investment approach. We show how to mitigate this negative effect by including long volatility funds in the investment portfolio. We also discuss adverse selection bias among investment managers, created by the investors demand for high unsustainable returns. Since one can only hope that the behaviour of either capital allocators or investment managers will change, we argue that it is left up to regulators to take measures to limit the use of credit and leverage, and to control the self-enforcing mechanism of market destabilization.

[1] Fusion Asset Management, 29 Queen Anne's Gate, London SW1H 9BU, UK; e-mail: kirill.ilinski@fusionam.com

[2] Laboratory of Quantum Networks, Institute for Physics, St-Petersburg State University, Ulyanovskaya 1, St.Petersburg 198504, Russia; e-mail: alexis.pokrovski@mail.ru

1 Introduction: Capital Line and Separation Theorem

For us the real attraction and the secret of success of the Capital Asset Pricing Model (CAPM) [1], the Arbitrage Pricing Theory (APT) [2] and no-arbitrage pricing methods, exists in the fact that these models describe the activities of economic agents, without modelling the agents themselves. Making sense of human choice for a risk/return profile is notoriously difficult, not only because of the peculiarities of humans perception of risk and reward [3], but also due to the sheer multi-dimensionality of the problem. Whether rational or irrational, humans don't make choices in isolation, but rather in the context of the investment universe available to them and within the realities of money management and social pressure.

This explains why the trick of substituting the complexities of choice between a manifold of investment portfolios [4] with a simple choice of leveraged unique optimal investment portfolio was always popular. To find unique and self-consistent price, investors do not have to trouble themselves anymore with aggregating demand and supply for the securities resulting from a host of individual choices of investment portfolios. Instead the pricing of the securities is now defined by their relationship with a single optimal market portfolio (CAPM) or set of market factors (APT). This leaves investors to decide on the relative allocation of capital between riskless assets (selected risky asset in zero-beta CAPM [5]) and the optimal portfolio. It is the gist of any *Separation Theorem - the complex and convoluted problem of investor choice and asset pricing is separated into two (almost) independent problems: the choice of the investor's risk profile and pricing of underlying securities* leading to the optimal investment portfolio in a risk-neutral world.

In laymans terms this means that as far as the investor is concerned, one has to find investment opportunities with an attractive ratio of expected return to some measure of risk (either standard deviation or more exotic tail-linked quantity [6]) and leverage up their investment to achieve required rate of return. All such investments are located on *the Capital Line - the line linking "riskless" asset and the optimal portfolio on the plane of risk and return.*

The argument of the Capital Line only works for ideal markets. As soon as the lending and borrowing rates are different or the transaction and stock borrow costs are introduced, a single optimal portfolio is smeared alone the efficient frontier. From a practical point of view, however, it hardly ever matters because the sizes of the corrections are typically quite small. Interestingly enough the corrections are counter-cyclical. During the market cycle downturn, when differences in borrow and lending is high, the credit (leverage) is expensive which precludes investors from leveraging up their positions. This makes the corrections relatively small. By the same token, during prolonged market upturn the price of leverage comes down together with rates bid/offer thus, once again, reducing the portfolio corrections.

There is, however, another very important factor in the investor behaviour which breaks down the Capital Line construction, produces a new bias in portfolio selection and violates the Separation Theorem. The factor is the sensitivity of an investor to a capital loss and the effect of stop-loss on an investment strategy. We argue here that when the effect of leverage is studied, the economic price of stop-loss brings non-linear (and often non-monotonous) dependence of return on the corresponding risk factor. On its own, this already leads to a manifold of optimal portfolios, depending on the level of risk assumed by an investor. The investor is no longer a faceless agent of CAPM with a common investment portfolio but a Markowitz's agent facing a choice of optimal investment to reach their investment objectives. Market portfolio looses its value as a benchmark and a market factor, beta has limited value to assets risk, asset pricing becomes once again a result of demand-supply dynamics. Here portfolio construction comes back as a value-adding exercise.

More interestingly, the analysis provides an intuition for a study of market mechanisms behind historical and recent market failures, as well as sheds light on psychological and behavioural biases among the investment community responsible for these failures. These effects, we believe, are general and do not depend on specific details of an asset pricing model. Therefore, to highlight the main effects and their implications here we deal with the simplest CAPM-type model, ignoring such technical complications as multiple time horizons, market friction, multiple factors, market inefficiencies and behavioural complexities of decision making.

1.1 Mark-To-Market loss and non-linearity of Capital Lines

Stop-loss is ubiquitous in trading and investing. Although it sounds like a personal choice or self-imposed element of discipline, in reality it is a rule of the game rather than an element of a strategy to succeed in the game. One can observe this on both micro- and macro-levels.

The micro level is very familiar to everyone. Margin trading imposes the natural stop-loss of loosing the full margin. Moreover, investors cannot afford to loose all their assets because if someone is playing a statistical game (as most of investment strategies are) one needs to "stay in the casino", preserving their assets to continue the game in case of any adverse streaks. Traders employing strategies based on mean-reversion need to apply stops to avoid being "cleared out" when the mechanism behind the mean-reversion temporarily breaks down.

On a macro level, considerable peak-to-trough losses cause clients to pull out of investment managers and investment styles. The redemptions lead to position liquidations and, in effect, loss-related forced selling. Another common source of loss-related position liquidation is connected with a partial reduction or even full withdrawal of trading and credit lines after peak-to-trough loss of 20-25% or similar loss during particular calendar period (quarter, year are typical period for ISDA agreements covering OTC derivatives agreements). Therefore, relative value strategies have a natural stop-loss enforced on them, either by investors or by trading counterparties.

It is important to emphasize that, while accounting for stop-loss in Wiener-type price process results in non-Wiener stochastic process for the position (or strategy) price, the dynamics of the price itself is not affected in any way. In other words, the stop-loss introduces a difference between the asset price process and the position price process, which exists only for the investor's viewpoint, changing respective investor's expectations of return and risk.

The stop-loss is accompanied by two effects. The first is the loss-locking, which creates a singularity on the left wing of the multi-modal return distribution function, in stark contrast to the assumed distribution function underlying most trading strategies. This effect, on its own, already brings non-linear dependence between risk and return which results in non-linearity of the Capital Line. The second effect is the increasing cost of position unwinding which contributes to further non-linearity of the Capital Line. It also causes Capital Lines to decrease for extreme levels of applied leverage.

In this paper we examine, in detail, the qualitative causes and effects of a concave Capital Line and review their consequences for investment management. The main results can be simply summarized as follows: **in the world of strongly concave and non-monotonous Capital Lines one cannot achieve higher and higher returns by taking higher and higher risks**. Pushing up the return targets results in increasing risk but not in increasing return due to decreasing and eventually negative marginal returns per unit risk. Thus, by chasing unrealistic returns and prompting investment managers to deliver unsustainable performances the investment community damages its own chances in a greedy search for yield and investment miracles. Besides negatively skewing the risk-return characteristics, this also enhances pro-cyclical dynamics and increases the component of volatility, not accompanied by the corresponding return.

The paper is organized as follows. In Section 2 we examine the effects of stop-loss in the most basic setup of modern portfolio theory and demonstrate how these lead to non-linearity of the Capital Line. Furthermore we introduce the negative impact of the price of liquidity and study the resulting negative slope of the Capital Line. The latter has profound consequences for investment management and economic policy making, which are considered in sections 3 and 4. Section 4 specifically examines the influence of the non-linearity of the Capital Line on cyclical volatility of capital markets. We conclude the paper with section 5 where we examine optimal target rate of return and its connection with assumed leverage. Derivation of formulas and other mathematical details are collected in the Appendix to avoid unnecessary disruption of the exposition in the main text.

2 Broken Capital Line

2.1 Constructing the Capital Line

When textbooks discuss the Capital Line they usually introduce:

1. A particular investment horizon T;
2. Opportunity set, a set of all possible portfolios with different weights of all admissible risky investment instruments but the same total initial capital allocation;
3. Some measure of potential loss during the period T which we call here risk or σ: this can be either standard deviation, square root of downside semi-variance, expected loss or any other quantity linear on leverage [6];
4. Existence of a "riskless" asset with term return r_0, i.e. the asset delivering return r_0 at time T irrespective of a particular realization of state variables, ex. irrespective of returns of risky instruments in the portfolio.

The "riskless" asset and the opportunity set are represented as points on the 2-dimensional plane of expected return $E(r)$ and σ (Figure 1), with the efficient frontier F. The "riskless" asset by definition has zero measure of risk and therefore has coordinates $(0,r_0)$. Other points form a convex set with the boundary F. In these notations the **Separation Theorem states that optimal portfolios, i.e. portfolios with maximal return for the corresponding risk σ, lie on the straight line** (the Capital Line CL, Figure 1) **which is tangent to the frontier F** at point O and crossing axis $E(r)$ at point $(0,\, r_0)$. In simple terms, if investor decided on having risk equal to σ, to achieve maximal expected return on the investment, then she has to have σ/σ_O fraction of her capital in the portfolio O financed by the combination of all of her initial capital and the capital received from buying $(1-\sigma/\sigma_O)$ fraction of the riskless asset. All portfolios on the Capital Line consist of the "riskless" asset

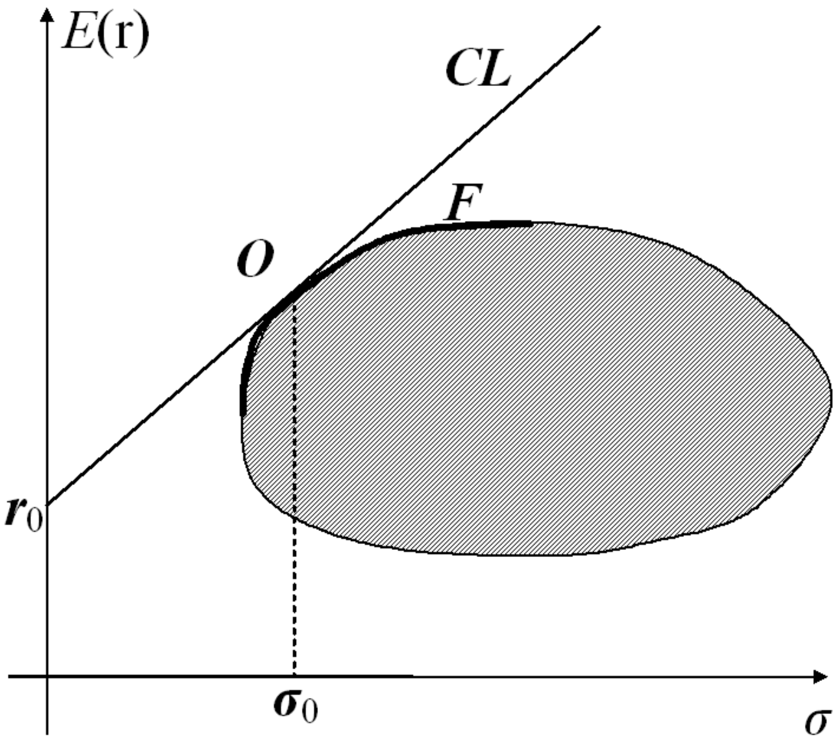

Figure 1: Markowitz Efficient Frontier F, Optimal Portfolio O and the Capital Market Line CL.

and the portfolio O so that whatever the particular risk parameter an investor has, it is optimal to have all risky investments in proportions of the portfolio O. This makes O the market portfolio.

What is important for us here is that, as soon as investors have access to the "riskless" asset and they have an ability to borrow and lend money at the "riskless" term rate r_0, they all have the same structure of their risky investments. This is quite different from the case when investors had to choose the value of their risk parameter first and only then pick up the corresponding point on the frontier F. This would lead to the situation when investors with different risk profiles would have different risky portfolios. These risky portfolios would then have to be aggregated with the corresponding weights to generate total demand-supply functions. The corresponding equilibrium prices which would make the solution of the problem self-consistent can then be found. Instead, if the Separation Theorem holds, one has to concern oneself only with the relationship between market portfolio O and the "riskless" asset, all this is due to the linear nature of the Capital Line.

In a more general situation the Separation Theorem means that if one finds the optimal structure for the investment and decides on the acceptable value of the risk parameter σ, the way to achieve maximal return is to leverage up the investment. In the world of Capital Line investment advisors don't have to think about changing the investment strategy if the risk parameters of their clients are different - one risky investment fits all. The only question is where to borrow money or, rather, where to sell the "riskless" assets short.

2.2 Stop-Loss

The situation changes dramatically if one adds the stop-loss condition. This can be demonstrated in a simple model we consider below, although the main results remain valid for a much wider range of models.

Let's assume that the price process for the optimal portfolio O can be described as a geometric Brownian motion with value S_0 at time $t = 0$. The corresponding return process $x = \ln \frac{S_t}{S_0}$ then follows the Wiener process with some volatility $\sigma \equiv \sigma_O$ and expected rate of return $\mu \equiv r_O$. Suppose also that at $t = 0$ an investor has amount of capital M_0, which she wants to employ with leverage L to buy $n = \frac{LM_0}{S_0}$ units of the optimal portfolio O. In what follows we re-scale prices so that $n = 1$ which therefore makes the leverage equal to $L = S_0/M_0$.

We introduce the stop-loss strategy as a condition that the maximum loss is bounded by the share $b \in (0, 1]$ of the total invested amount M_0. The stop-loss imposes the exit condition for $M_t = bM_0$, which restricts the stochastic dynamics by the semi-line $x > x_L \equiv \ln(1-b/L)$, $x_L < 0$. When $x = x_L$, the dynamics stops and the trade is unwound at residual price.

Due to the stop-loss the probability distribution of the return is strongly asymmetric and requires careful definition of the downside risk. In this article we take the square root of the doubled downside semi-variance as a measure of risk:

$$\sigma_<^2/2 = E\{(m_t - E\{m_t\})^2 | m_t < E\{m_t\}\},$$

where $m_t = M_t/M_0$. The condition $m_t < E\{m_t\}$ may be changed to $m_t < m_{max}$, where m_{max} is a level of loss which corresponds to a particular level of VAR. This would make the results even more pronounced.

The simplicity of the model chosen here allows one to derive explicit results for the distribution function in presence of the stop-loss (see Appendix). The main result is, however, so simple that it can be easily demonstrated graphically (Figure 2).

The example we use here assumes that the optimal portfolio without the stop-loss has average return of $\mu = 40\%$ and the standard deviation $\sigma = 55\%$. This (approximately) corresponds to the optimal portfolio for a simplest two-asset portfolio in a zero interest rate environment which we consider as a base case later in the paper. The first asset has

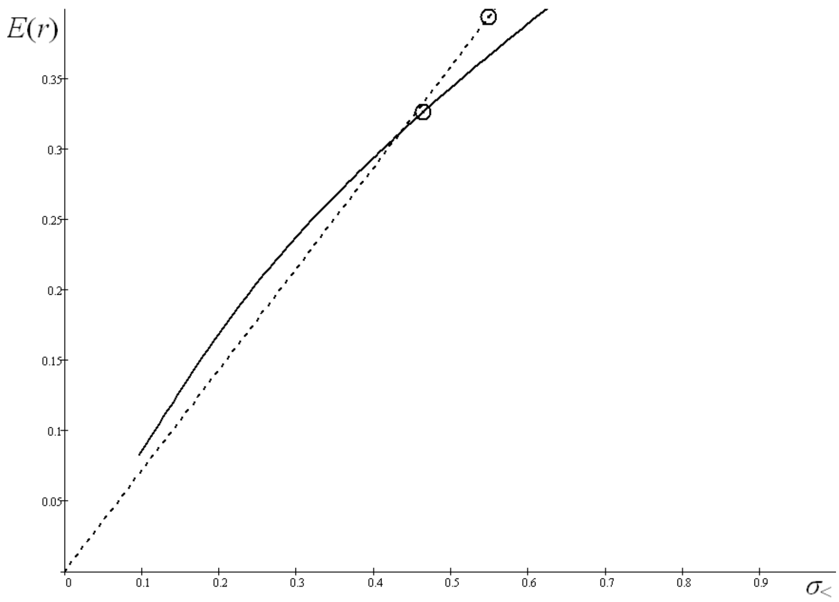

Figure 2: Underlying asset with average return $\mu = 40\%$, and risk $\sigma = 55\%$. Time horizon 1 year, riskless interest rate is taken equal to zero. Dashed line: Capital Market Line, assuming no stop-loss. Solid line: non-linear deformation of the Capital Line, assuming stop-loss at 20%. Circles correspond to portfolios with leverage equal to 1.

instantaneous average return $\mu_1 = 25\%$ and standard deviation $\sigma_1 = 40\%$. The second asset has average return $\mu_2 = 50\%$ and standard deviation $\sigma_2 = 80\%$. We assume here correlation $\rho = 40\%$.

Figure 2 demonstrates the effect of the stop-loss on the Capital Line: what used to be linear becomes curved. This reflects the fact that taking higher and higher risk by taking higher leverage is not fully compensated by the corresponding return. High leverage increases the probability of touching the stop-loss level, which stops the game and does not allow investments to recover. This has disproportional effect on the returns as the marginal return becomes a diminishing function of the risk.

Another way to see that the returns are not linear in risk anymore is to plot the marginal return on risk (Figure 3). The reader can see that the rising return target brings a deterioration of the risk/return profile with a strongly diminishing margin return, i.e. every new unit of assumed risk brings smaller and smaller benefit of return.

2.3 Stop-Loss and the price of liquidity.

The next step is to add the price of liquidity to the model. We introduce it as a price of the position unwind. The price is proportional to the size of the position and therefore is proportional to the leverage factor. Once again, the Appendix provides exact results for the distribution function and risk-return characteristics in our simple model for the case of stop-loss with price of liquidity.

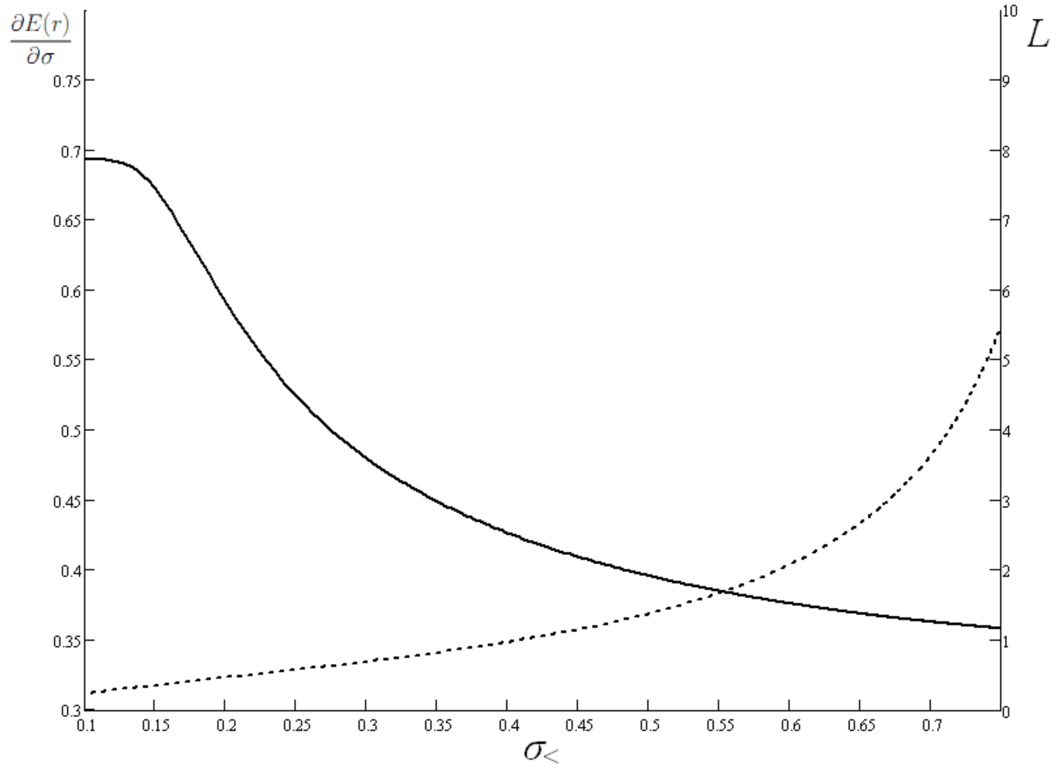

Figure 3: Solid line: Marginal expected return $\frac{\partial E(r)}{\partial \sigma}$ per unit risk (left Y-axis). Dashed line: leverage L versus $\sigma_<$ (right Y-axis). The parameters of the underlying asset and stop-loss as in Fig. 2.

In the presence of the price of liquidity any excessive leverage contributes adversely to the expected return and makes the marginal return to risk negative for sufficiently large leverage. This can be seen on Figure 4 and 6, which are analogues of Figures 2 and 3 for the case of additional price of liquidity. We use the same parameters as in the example of Figure 4 but we also assume that the position unwind incurs losses equal to 5% of the position size.

One can see that high levels of risk and leverage are accompanied by decreasing expected returns, which results in worsening risk/return profile and, eventually, a negative utility for an investor. The lower line is non-monotonous and reaches its peak at around 38% return and risk 81%, after which the return starts to decrease. Any additional unit of risk brings negative return contribution. This can be seen from Figure 6 which demonstrates a negative marginal return on risk for large leverage factors.

Figure 7 shows the expected return, the downside variance and the probability to hit the stop-loss as a functions of leverage for the same value of parameters.

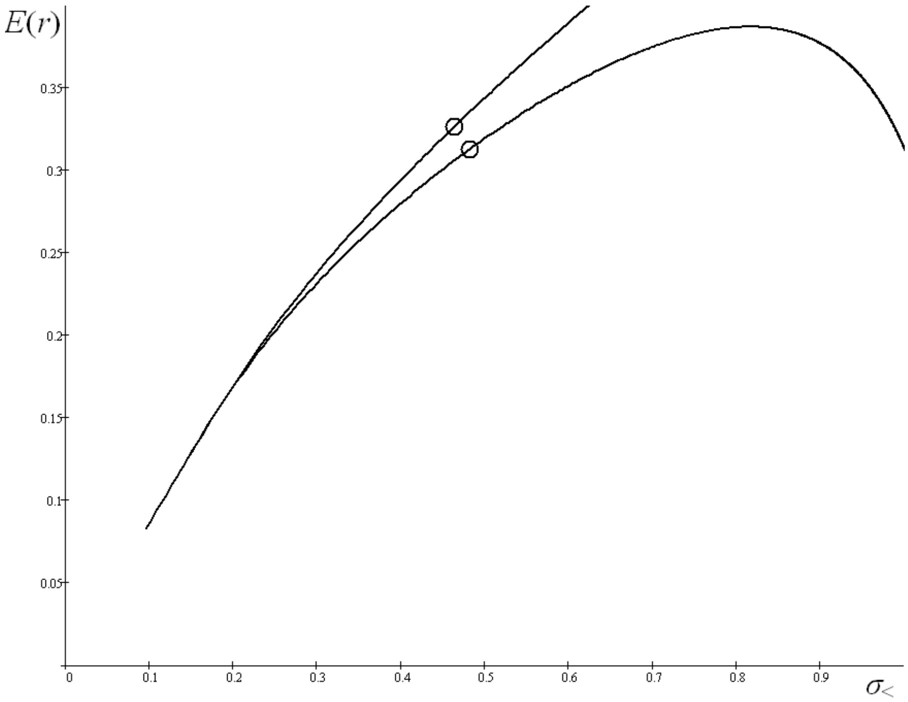

Figure 4: Effect of price of liquidity on Capital Line. Same parameters as in Fig. 2, assuming liquidity premium $\alpha = 5\%$. Upper line: Expected return versus risk, stop-loss but no liquidity premium. Lower line: Expected return versus risk, assuming liquidity risk. Circles correspond to leverage 1. Maximal achievable return is 38% with downside volatility of 81%.

3 Implications for investment management

Let us return back to our example of two-asset portfolio with instantaneous parameters $\mu_1 = 25\%$, $\mu_2 = 50\%$, standard deviations $\sigma_1 = 40\%$, $\sigma_2 = 80\%$ and correlation $\rho = 40\%$ in a zero interest rate environment.

3.1 Envelope Line

In world of non-monotonous Capital Lines there exists a new interesting effect. To achieve a maximal return one has to take leverage sub-optimal portfolio, not a portfolio in which Capital Line touches Markowitz Efficient Frontier. This is because it is beneficial to accept lower return (sub-optimal portfolio) which is accompanied by lower leverage and, hence, less frequent stop-loss event and lower liquidity charge for the same value of risk. Maximal achievable returns for all risk parameters are now located on the curve which is enveloping all Capital Lines piercing the Efficient Frontier. We call this line the **Envelope Line** EL $(\sigma, E(r))$. In the case of stop-loss and price of liquidity the Envelope Line plays the same role the Capital Line played in the idealized CAPM model.

This is shown in more details on Figure 5.

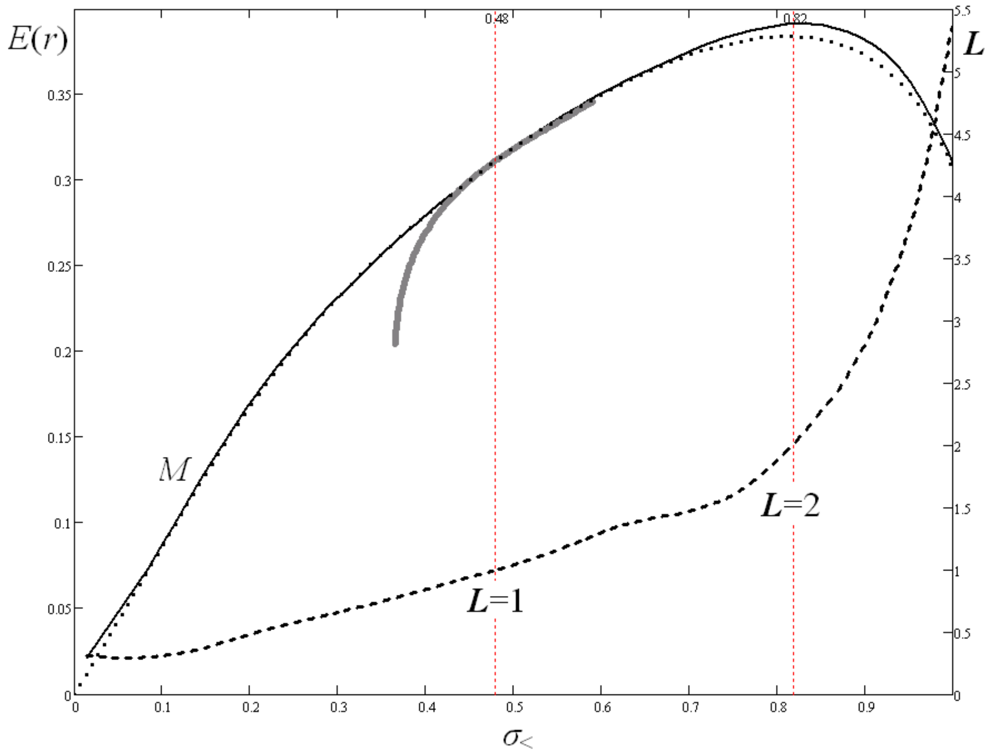

Figure 5: Thick grey line: Markowitz boundary for two-asset portfolio. Instantaneous parameters $\mu_1 = 25\%, \sigma_1 = 40\%, \mu_2 = 50\%, \sigma_2 = 80\%, \rho = 40\%$.
Solid line: Enveloping Line for the Markowitz boundary. Dotted line: the Capital line tangential to the Markowitz boundary. Tangential point at $\mu = 31.2\%$, $\sigma_< = 48.2\%$.
Dashed line: leverage for the Envelope Line (right Y-axis) versus $\sigma_<$.
Liquidity premium $\alpha = 5\%$. Time horizon 1 year.

The figure shows two lines, solid and dotted. The dotted line (it is the lower line on Fig. 4) is the Capital Line, tangential to the Markowitz Efficient Frontier. This line is an evolution of the Capital Line of CAPM under the stop-loss and non-zero liquidity price. The solid line is built as a line enveloping all Capital Lines crossing the Efficient Frontier in all its points. One can see that the lines do not coincide and the deviation is maximal at the point of maximal return. It means that, due to the non-monotonous character of the Capital Lines, to achieve different returns at minimal risk one indeed has to select different risky portfolios to leverage up. In other words, because of the non-linear dependence of the risk measure on leverage and price of liquidity, an investor is better off by selecting sub-optimal portfolio (in CAPM sense), which is safer to leverage up, to achieve required return at minimal risk.

3.2 New effects of the Envelope Line

The concavity and non-monotonous nature of the Envelope Line is a qualitatively new effect which leads to several important observations:

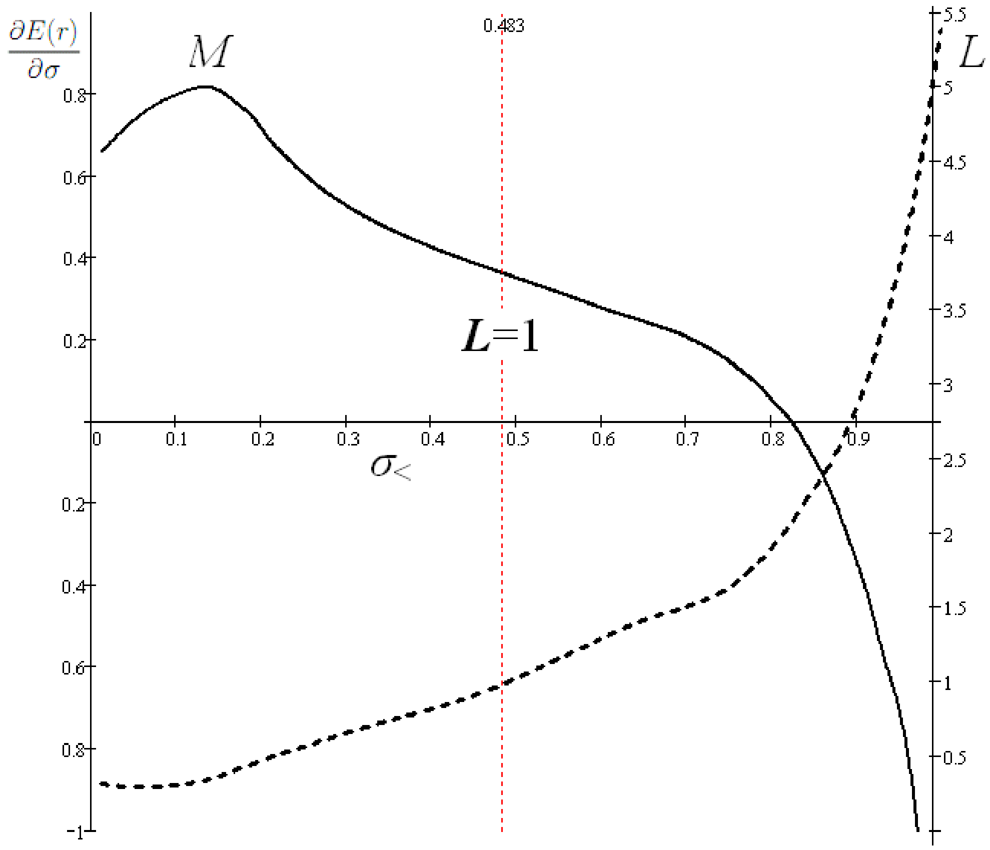

Figure 6: Solid line: Marginal expected return per unit risk $\frac{\partial E(r)}{\partial \sigma}$ (left Y-axis) for the Envelope Line of Fig. 5 versus $\sigma_<$. Dashed line: leverage L (right Y-axis) for the Envelope Line versus $\sigma_<$. M denotes the point where marginal return to risk is maximal.

1. The Separation Theorem is dead. If points on the Envelope Line correspond to different risky portfolios O', there is no unique market portfolio O as in the case of the Capital Line. Composition of risky portfolios O' is dictated by investor preferences, demanded return and risk/return profiles and are results of the aggregated demand. The prices of the underlying securities are driven by matched demand and supply, and are a result of the self-consistent solution of a multi-agent problem.

2. There exists a maximal achievable expected return. Investors cannot push for return too much as this leads to excessive leverage and negative marginal return to risk. There is a maximal expected return which corresponds to the optimal leverage ($L = 2$ on Figure 5) beyond which investors are simply diminishing their returns while increasing their risk.

3. The best risk/return profile is not at the point of highest achievable return. It is apparent from Figure 6 that the point of maximal return and optimal leverage is located on the concave part of the Envelope Line. At this point the marginal return to risk is zero (Figure 6). One can argue that the best risk/return profile is achieved at the point M, which has highest marginal return to risk under full capital allocation but lower return comparing to the return at the point of maximal return, where $\frac{\partial E(r)}{\partial \sigma} = 0$. On the other hand, lower

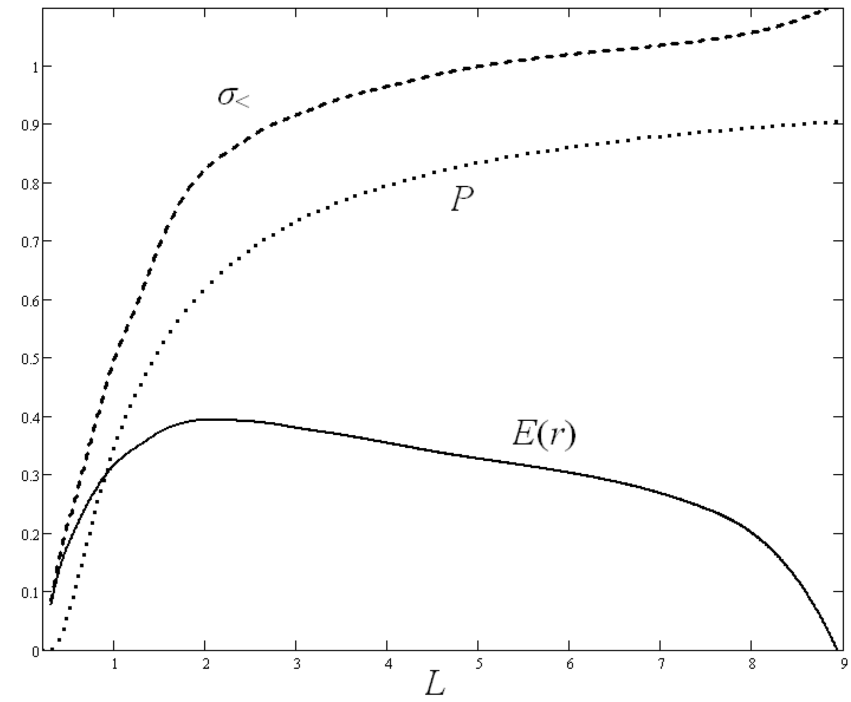

Figure 7: Expected return $E(r)$ (solid line), downside risk $\sigma_<$ (dashed line) and the probability P to hit the stop-loss level (dot line) as functions of leverage L for the Envelope Line of Fig.5

quality of returns is the price which investors have to pay to achieve a higher level of expected returns.

3.3 Implications for arbitrage trading

The notion of arbitrage is tricky and often misleading. Even in theory, arbitrage strategies are not as straightforward as textbooks depict them. Traditionally, arbitrage is defined as a situation when one can achieve positive return which is higher than risk-free return (to allow leveraging up) by simultaneous buying and selling financial instruments to offset the corresponding risks. Since the instruments which are bought and sold are not identical, the operation has an intrinsic mark-to-market risk which is the increasing function of leverage and is subject to the stop-loss and the price of liquidity. The latter is even more important here because of the large trading sizes which arbitrageurs need to transact to achieve required rate of return from typically small arbitrage returns.

The fact that many players run copycat strategies and "feel the pain" at the same time, creates a coherent effect of unwind which drains the liquidity and makes the price of liquidity at stop-loss (the price of liquidation) particularly high. Managers get a false sense of security from the low price of liquidity during quiet times when the trades are put on, subsequently underestimating the liquidity effect caused by the forced unwind. It is also often the case that the account of the potential liquidation and the associated liquidity penalty, would make the trades unattractive in the first place. In the search for yield, managers ignore the "Mexican peso" effect of the historically unobservable liquidity-caused jump and initiate relative value trades with intrinsically negative value. The events of 2007-2009 demonstrated this effect for several arbitrage strategies including credit basis, credit curve trading, convertible arbitrage, dividend arbitrage and volatility arbitrage.

Applying the same logic we advocated above to the arbitrage trading reveals several interesting consequences:

a In practice non-leveraged arbitrage return is often so small that to match the investors expectations, investment managers-arbitrageurs have to push leverage up so much that the expected return for investors becomes negative.

b Arbitrage should persist if the non-leverage arbitrage return is so small, that even after leveraging up the expected return will still be smaller that the required rate of return. We call this arbitrage the virtual arbitrage and consider it and its effects on pricing in some details in [7]. Trading insufficient arbitrage returns indicates "squeeze for yield" and excess liquidity in the market which can be monitored by both policy-makers and market participants.

c Many arbitrage strategies employ illiquid instruments, either to finance liquid positions or to extract liquidity premium outright. Basis, curves, converts, small cap are examples of such trades. Illiquidity of a portfolio is another type of leverage if measured as a ratio of the total size of positions to immediately available capital. Therefore it incurs the same risks and leverage costs as a portfolio of liquid leveraged instruments. Liquidity mis-match recognized as leverage for banks and financial intermediaries, is not yet treated as such in case of "buy-side" firms and skews the risk-return analysis of trading styles. The issue was touched on in Ref [8].

d Optimal arbitrage strategies have to take into account mark-to-market risk, liquidity pricing and the stop-loss before evaluating their benefits (see [9] on optimal arbitrage trading under mark-to-market risk). Often the arbitrage strategies are risky and expensive substitutes for common liquidity-providing strategies and are mis-sold to investors under the clock of "quant money machines".

e Arbitrage strategies are particularly sensitive to liquidity pricing and therefore have hidden costs which are rising with market volatility. This shifts the risk/return profile against an investor and clearly demonstrates that the arbitrage strategies are short volatility even if the risk parameters ("greeks") shown to an investor are not reflecting this.

4 Pro-cyclical nature of leverage investing and its implications for economic policy-making

Non-linearity of leverage investing covered in the last section poses several important questions related to global financial stability and social benefits.

4.1 De-stabilizing effect of optimal leverage and market instability

One of the most important consequences of the considered effects is the pro-cyclical nature of the optimal leveraging and the leverage-related losses.

So far we have concentrated on the static profile (Figure 5) when market parameters, including volatilities and correlations, remain constant. If long-term market dynamics are taken into account, one has to investigate the effects of simultaneous changes in the instruments volatility and correlations. We illustrate these effects on the example of the market which consists of two securities with expected returns of 20% and 30%, initial volatilities of 30% and 40% and correlation 30%, although results are qualitatively the same for the more general case of multi-instrument markets. Figure 8 depicts the Envelope Line for this case. We then simulate the effect of market correction by raising volatilities by 20% and increasing correlation from 30% to 60%. Figure 9 shows the position of the Envelope Line in the case of high volatility and high correlation environment, which is typical for market sell-offs. Figures 8 and 9 demonstrate an interesting effect: rising volatilities and correlations cause the optimal leverage point (L=3.4, Fig. 8) to become sub-optimal and over-leveraged while the true optimal leverage is reduced considerably (L=1.6, Fig. 9). This forces an investor to deleverage by reducing the excess exposure in an adverse market environment. The opposite shift is observed when volatilities and correlations are subsiding and the optimal leverage point (L=1.6 Fig. 9) becomes under-leveraged (see L=1.6, Fig. 8). In a quest to maintain the optimal leverage/maximal return positioning, an investor has to increase his exposure in a typically low-yield environment, which accompanies low volatility/low correlation and, therefore, low risk premium regimes. In simple terms, to maintain an optimal leverage position, the investor has to sell when the market is going down and has to buy when the market is going up, in a typical "negative gamma", short-option trading profile. This buying and selling occurs simultaneously for all investors thus synchronizing market activity, negatively affecting the price of liquidity, exacerbating oscillations and creating instability.

It is worth noting that a similar "negative gamma" effect caused by dynamical changes in leverage, was recently examined by Adrian and Shin in Refs [10],[11]. The authors studied changes in the balance sheets of financial intermediaries and found a strong pro-cyclical dependence of financial leverage leading to self-enforcing market oscillations, similar to the ones described above. Our research is complementary in the sense that we introduced a mechanism of "negative gamma" generated by "buy-side" firms, while Adrian and Shin have found the mechanism on the "sell-side". In both cases the effect is caused by a search for yield during benign markets and forced unwind during market deterioration. Our consideration of stop-loss and the price of liquidity can be seen as an attempt to put a price on an event risk of loss spiral and margin spiral, which were described by Brunnermeier in [12] as

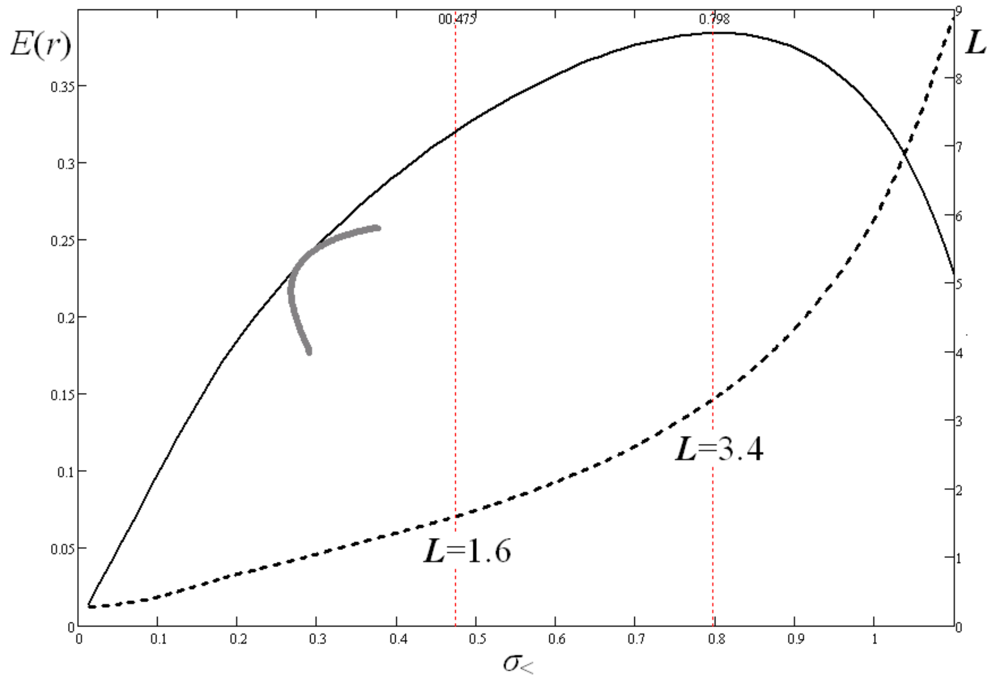

Figure 8: Markowitz boundary (thick grey line), Envelope Line (solid) and leverage (dashed line) for two-asset portfolio with instantaneous parameters $\mu_1 = 20\%, \sigma_1 = 30\%, \mu_2 = 30\%, \sigma_2 = 40\%, \rho = 30\%$.

mechanisms for amplifying market instability. In the context of hedge fund performance, the dynamical dependence of leverage on volatility environment and the "negative gamma" effect was observed many times, mainly based on anecdotal evidence. McGuire and co-authors in Refs [13], [14] developed the first quantitative framework to systematically estimate the funds leverage. To this end they used factor analysis of funds returns with a choice of linear and non-linear market factors.

4.2 Short volatility profile of portfolio construction and long volatility funds as way to correct this

To correct the observed behaviour, an investor has to realize that the strategy of maintaining optimal leverage in the search for yield is intrinsically short market volatility, or short vega and short gamma in option parlour. One can suggest two possible ways to correct the undesirable behaviour. 1: Either constantly under-leveraging its portfolio (effectively delta-hedging the underlying short optionality and under-performing competitors) 2: Adding long-volatility investments with positive vega and positive gamma to the portfolio profile. In both cases the price of the hedge will be an underperformance of the combined portfolio in low volatility environment comparing with unhedged portfolios. This has to be expected because investors, effectively, enhance their performance by selling the hidden option, ingrained into their portfolio construction.

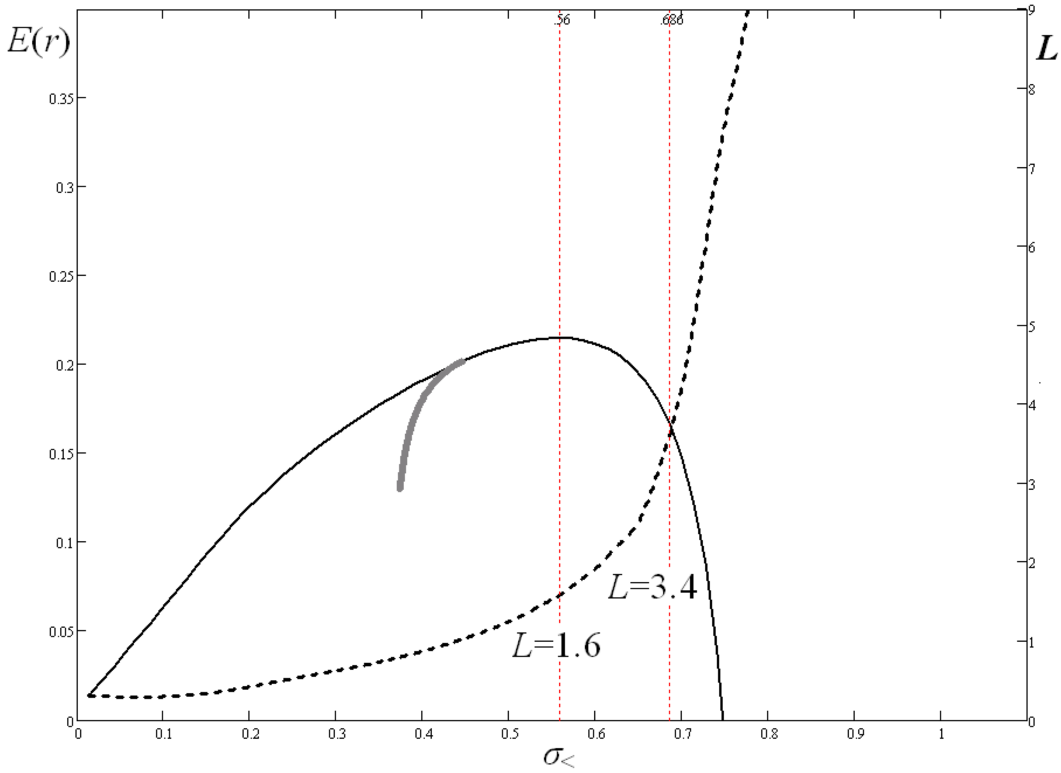

Figure 9: Markowitz boundary (thick grey line), Envelope Line (solid) and leverage (dashed line) for two-asset portfolio with instantaneous parameters $\mu_1 = 20\%, \sigma_1 = 50\%, \mu_2 = 30\%, \sigma_2 = 60\%, \rho = 60\%$.

The inclusion of long volatility instruments or a long volatility fund, does not completely eliminate the problem of moving optimal leverage, but still considerably reduces the amplitude of the change and more importantly, almost negate the negative effect on the expected return. Due to the relative flatness of the Envelope Line around its optimal leverage point, one does not have to have a tight control on leverage if one is ready to accept small sub-optimality for the expected return on the portfolio. Figures 10 and 11 demonstrate how the addition of a long volatility fund to the two-securities markets of Figures 8 and 9 change the investment profile, considerably reduce sensitivity for the expected return to volatility regimes and dampens the instability enhancement mechanism. For simplicity the volatility fund is modelled here as a variance swap position of $\$500K$ vega for $\$100mln$ optimal investment portfolio of Fig. 8 with initial optimal leverage $L = 3.4$. Real long volatility funds out-perform simple variance swaps by adding value through selecting assets with higher potential for volatility spikes and by mitigating variance premium through utilizing sophisticated delta-hedging strategies to cover the time decay on a fund's long options positions.

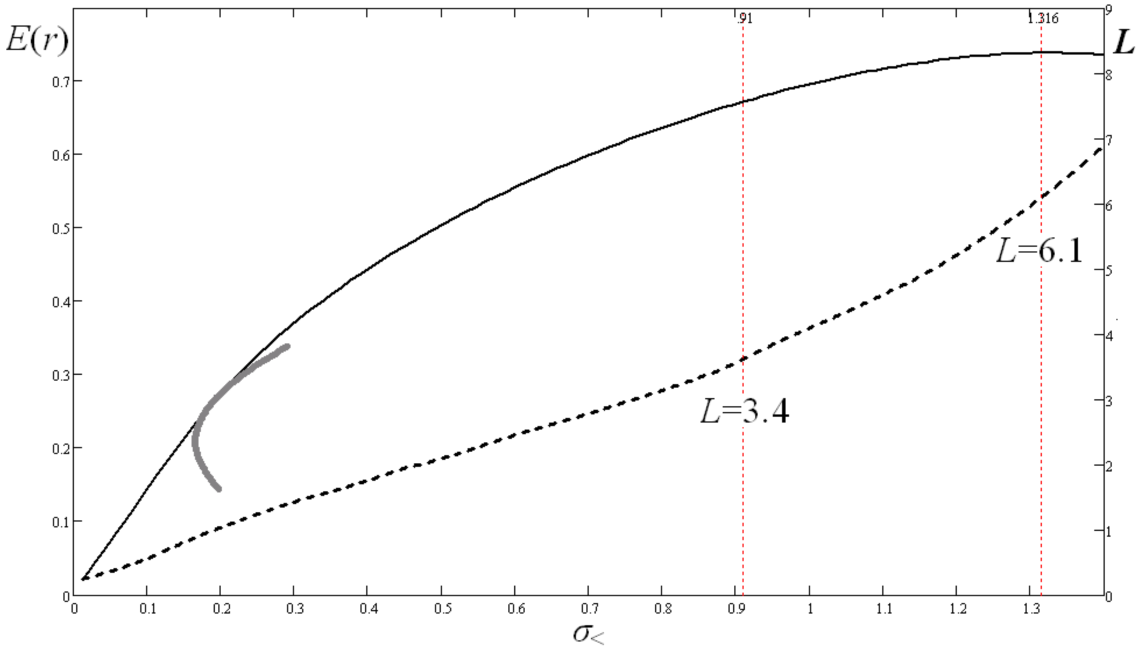

Figure 10: Markowitz boundary (thick grey line), Envelope Line (solid) and leverage (dashed line) for two-asset portfolio and a variance swap: $\mu_1 = 20\%, \sigma_1 = 20\%, \mu_2 = 30\%, \sigma_2 = 30\%, \rho = 10\%$, vega of variance swap is equal to 0.5% of the investment notional and strike is 10 volatility points higher.

4.3 Implications for economic policy making

The discussion of the pro-cyclical nature of optimal leverage investment and its de-stabilizing effect, poses a question about possible regulatory measures to mitigate the undesirable economic instability.

The push for yield and excessive leverage is driven by investors demand for high and unsustainable returns. Chasing returns, as documented for example in Ref. [15], creates a characteristic adverse selection among investment managers. This is because rather than allocating capital towards investment managers with high quality returns, capital is allocated to investment managers with sub-optimal investment profiles but who target higher returns by taking an unjustifiably high risk. This forces originally prudent managers to compete for capital by changing their investment strategy to generate higher returns at the expense of a deteriorating risk/return profile, thus joining the crowd of yield-chasers. Observing the history of credit and leverage cycles [16] as well as recent events, one can hardly hope that the behaviour of either capital allocators or investment managers will change anytime soon, therefore re-creating the mechanism for market instability over and over again. In this situation it should be, in our opinion, left up to regulators to take measures that establish limits on the use of leverage and to control the self-enforcing mechanism of market destabilization. As authors of the recent article [16] put it "credit growth is a powerful predictor of financial crises, suggesting that such crises are credit booms gone wrong and that policymakers ignore

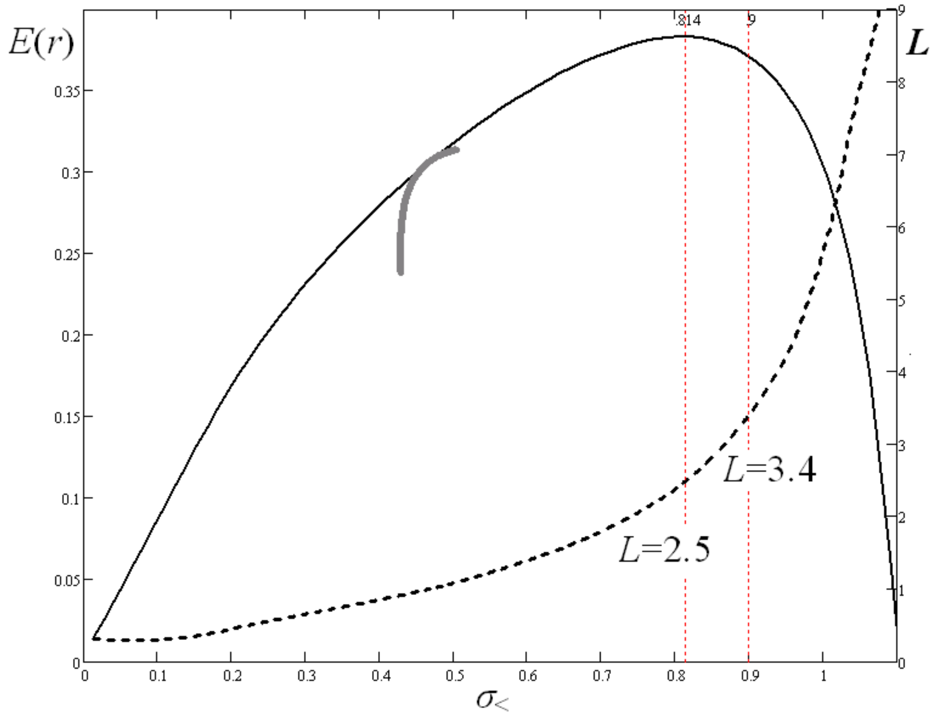

Figure 11: Markowitz boundary (thick grey line), Envelope Line (solid) and leverage (dashed line) for two-asset portfolio and a variance swap: $\mu_1 = 20\%, \sigma_1 = 50\%, \mu_2 = 30\%, \sigma_2 = 60\%, \rho = 60\%$, vega of variance swap is equal to 0.5% of the investment notional and strike is 20 volatility points lower.

credit at their peril".

It is a challenging task to define and monitor optimal leverage as used by the market participants. It is, however, possible to observe the performance of certain arbitrage strategies and use their evolving risk/return characteristics as an indicator of market leverage. As we pointed out earlier, arbitrage should persist if the non-leverage arbitrage return is so small that after optimal leveraging up, the expected return is still smaller than the required rate of return. Utilizing the insufficient arbitrage returns indicates a "squeeze for yield" and excess liquidity in the market, which can then be tracked by regulators on a daily basis.

It is safe to assume that capital allocators will continue incentifying investment managers, to achieve the highest possible returns by leveraging up their portfolios. Therefore the role of regulators should be to identify optimal leverage points and to limit the use of leverage by raising either the cost of leverage or the corresponding capital charges (capital reserves against leveraged positions). We see the beginning of state-controlled clearing houses working together with central banks as the only way of bringing transparency to the use of leverage. This has to be coupled with creating an effective mechanism of control of the level of leverage by adjusting credit/margin requirements [3].

[3]Pro-cyclical requirements, as far as we know, were first discussed in Ref.[18]

5 Conclusion: fair level of return and price of liquidity option

If there is one conclusion investors should take from this article, it should be that the chasing of higher than long-term average returns can damage investors own prospects through the creation of higher volatility which is not accompanied by the corresponding expected return. An investor can ask for 30% annualized return only to find out later that the average long-term return is much lower, while the volatility during sharp corrections is business-threatening. So what should be this long-term average return which investors need to target to avoid boom-and-bust cycles, and how does it relate to the parameters of global economic activity? The answer to this question requires further research, but we venture a conjecture that the long-term average investment return is in the range of $5 - 15\%$ and it is related to the long-term average interest rates and real GDP growth which are fed through average dividend returns and their uncertainty. Table 1 illustrates (somewhat speculatively) this point by listing long-term average returns of high profile investment managers, investment styles, investment indices and real businesses.

This table allows readers to make several interesting observations. The first one is that various average long term returns are indeed within the range of $5\% - 15\%$. The second observation is that the skill of good managers is reflected, not in their absolute return or risk-return profile, but rather in their ability to limit drawdowns by active portfolio management. The third observation concerns the link between the use of leverage and performance characteristics of strategies. One can see that worst-performing styles, both in terms of return-to-volatility ratios as well as drawdown-to-return ratios are Fixed Income Arbitrage and Equity Market Neutral. This means that the strategies which use quantitative methods to identify a small relative value opportunities and then leverage up[4] the positions to achieve a desirable rate of return. At the same time, the best-performing style is Distressed which is typically a low leverage, cash-type investment strategy[5]. It is tempting to interpret the observation as supporting evidence for the hidden price of leverage as a liquidity and stop-loss option premium. The premium is received and remains unobservable during quiet periods and "kicks in" during hard periods of negative performance.

The results we see here are not completely new. In paper [17] Bertelli found that the use of leverage worsens the efficiency of hedge fund strategies measured by various ratios. He also found that fixed income arbitrage strategy performs better without leverage. Authors of Ref. [19] found no dependence of performance on leverage. This is probably due to them analyzing a relatively short period of time (2000-2003) when general hedge fund performance was positive. Yet, they also found that for strategies with negative sample performance (Equity Hedge, Merger Arbitrage and Equity Market Neutral) the correlation of Sharpe ratio with leverage is negative. This has to be expected in our model since the deterioration of the risk-return profile is explained by the stop-loss.

Summing up, we argue that wrong incentives lead to trading style herding, which, in

[4] "Good times" median leverage 2 for Equity Market Neutral [19] and 2-5 for fixed income strategies[17].
[5] "Good times" median leverage 1 [19].

Table 1: Long-term average returns of high profile investment managers

Assets	Average annualized return	Volatility (200 monthly)	The worst drawdown	Ratio of return to volatility	Ratio of drawdown to return
Berkshire Hathaway INC-CL A (BRK/A US EQUITY; 1990 -2009)	15,8%	21%	-44%	0,75	-2,79
Quantum Endowment Fund A1 (QUTQUAI NT EQUITY; 1992-2009)	9,7%	21%	-29%	0,46	-3,01
S&P500 (SPX INDEX; 1990-2009)	6,1%	15%	-52%	0,41	-8,52
Credit Suisse Tremont Hedge Fund Index (HEDGNAV INDEX; 1994-2009)	9,3%	8%	-19%	1,16	-2,05
Credit Suisse Tremont Hedge Fund Equity Neutral Market Index (HEDGNEUT INDEX; 1994-2009)	6%	13%	-44%	0,43	-7,89
Credit Suisse Tremont Hedge Fund Emerging Markets (HEDGEMGM INDEX; 1994-2009)	7,9%	16%	-32%	0,49	-4,06
Credit Suisse Tremont Hedge Fund Distressed (HEDGDIST INDEX; 1994-2009)	11%	7%	-22%	1,54	-2,04
Credit Suisse Tremont Hedge Fund Convertible Arbitrage (HEDGCONV INDEX; 1994-2009)	7,6%	8%	-33%	1,01	-4,36
Credit Suisse Tremont Hedge Fund Fixed Income Arbitrage (HEDGFIAR INDEX; 1994-2009)	5%	6%	-29%	0,75	-6,16
Credit Suisse Tremont Hedge Fund Long Short Equity (HEDGLSEQ INDEX; 1994-2009)	9,8%	10%	-22%	0,98	-2,24
Credit Suisse Tremont Hedge Fund Managed Futures (SCTFUTR INDEX; 1994-2009)	6,0%	12%	-17%	0,50	-2,85

turn, pushes managers to sell "tails". These then need to be leveraged up because the absolute price of the "tails" is small and getting squeezed even smaller by the herd (Ref. [18]). The leveraging up is equivalent to selling options on events of loss-forced unwinds, accompanied by liquidity hits, which are exacerbated further by the herding. Selling these options explicitly damages the investors risk-return profile and destabilizes the financial system. The situation can be corrected by either making a deliberate change in portfolio construction or by additional requirements and checks applied by regulators. There is no other way.

6 Appendix. Explicit formulae for the expected return and downside semi-variance

We assume here that the price process for the optimal portfolio O can be described as a geometric Brownian motion with value S_0 at time $t = 0$. The corresponding return process $x = \ln \frac{S_t}{S_0}$ therefore follows the Wiener process with some volatility $\sigma \equiv \sigma_O$ and expected rate of return $\mu \equiv r_O$. An investor at $t = 0$ has an amount of capital M_0, which they employ with leverage L to buy $n = \frac{LM_0}{S_0}$ units of the optimal portfolio O. In what follows we re-scale prices so that $n = 1$ which therefore makes the leverage equal to $L = S_0/M_0$.

We introduce the stop-loss strategy as a condition that the maximum loss is bounded by the share $b \in (0, 1]$ of the total invested amount M_0. The stop-loss imposes the exit condition for $M_t = bM_0$, which restricts the stochastic dynamics by the semi-line $x > x_L \equiv \ln(1-b/L)$, $x_L < 0$. When $x = x_L$, the dynamics stops and the trade is unwound at residual price (either with or without liquidity costs). In this article we take the square root of doubled downside semi-variance as a measure of risk:

$$\sigma^2_</2 = E\{(m_t - E\{m_t\})^2 | m_t < E\{m_t\}\},$$

where $m_t = M_t/M_0$. The formulae below can be easily changed for a more general case of $m_t < m_{max}$, where m_{max} is a level of loss which corresponds particular level of VAR.

The return of the investment is $m_t = M_t/M_0 - 1 = L(e^x - 1)$. The probability distribution $W_t(x)$ is highly assymmetric. Indeed,

$$dW_t = (P(t)\delta(x - x_L) + u(x, t))dx,$$

where $P(t)$ is the probability to touch the barrier within time interval $[0, t]$.

The evolution of the probability density is described by the Fokker-Plank equation

$$\frac{\partial u}{\partial t} = -\mu \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}$$

with the absorbing boundary condition $u = 0$ at $x = x_L$ [20], [21]. Solution of the equation with the initial condition $u(x, 0) = \delta(x)$ is

$$u(x, t) = \frac{1}{\sigma\sqrt{t}} \left\{ \phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{2\frac{\mu x_L}{\sigma^2}} \phi\left(\frac{x - \mu t - 2x_L}{\sigma\sqrt{t}}\right) \right\},$$

where $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. The probability density to touch the stop-loss level at time t is

$$p(t) = -\frac{x_L/\sigma}{t^{3/2}}\phi\left(\frac{\mu t - x_L}{\sigma\sqrt{t}}\right),$$

and

$$P(t) = \int_0^t p(s)ds = e^{2\frac{\mu x_L}{\sigma^2}}\Phi\left(\frac{-x_L - \mu t}{\sigma\sqrt{t}}\right) + \Phi\left(\frac{\mu t - x_L}{\sigma\sqrt{t}}\right),$$

where $\Phi(x) = \int_x^\infty \phi(s)ds$, $\Phi(-\infty) = 1$. The probability to hit stop-loss at all, at any time, is given by $P(\infty) = e^{2\frac{\mu x_L}{\sigma^2}} = (1 - b/L)^{2\frac{\mu}{\sigma^2}}$.

Expected return. If we introduce the following functions

$$f_0(s) = \int_s^\infty u(x,t)dx = \Phi\left(\frac{s - \mu t}{\sigma\sqrt{t}}\right) - e^{2\frac{\mu x_L}{\sigma^2}}\Phi\left(\frac{s - \mu t - 2x_L}{\sigma\sqrt{t}}\right) \quad ,$$

$$f_1(s) = \int_s^\infty e^x u(x,t)dx =$$

$$= e^{(\mu+\sigma^2/2)t}\left\{\Phi\left(\frac{s - (\mu + \sigma^2)t}{\sigma\sqrt{t}}\right) - e^{2\frac{\mu x_L}{\sigma^2} + 2x_L}\Phi\left(\frac{s - (\mu + \sigma^2)t - 2x_L}{\sigma\sqrt{t}}\right)\right\} \quad ,$$

$$f_2(s) = \int_s^\infty e^{2x}u(x,t)dx =$$

$$= e^{2(\mu+\sigma^2)t}\left\{\Phi\left(\frac{s - (\mu + 2\sigma^2)t}{\sigma\sqrt{t}}\right) - e^{2\frac{\mu x_L}{\sigma^2} + 4x_L}\Phi\left(\frac{s - (\mu + 2\sigma^2)t - 2x_L}{\sigma\sqrt{t}}\right)\right\} \quad .$$

The return at time t is $m_t = M_t/M_0 - 1 = L(S_t/S_0 - 1) = L(e^x - 1)$. The expected return is

$$E(m_t) = LE(e^x) - L = Lf_1(x_L) - (L-1)(1 - P(t)) + (1 - b)P(t)$$
$$= Lf_1(x_L) - (L - 1) + (L - b)P(t).$$

Downside semi-variance. Let's denote $\bar{x} = \ln(1 + \frac{E(m_t)}{L})$, so that $E(m_t) = L(e^{\bar{x}} - 1)$. We have

$$\sigma_<^2/2 = E\{(e^x - e^{\bar{x}})^2 | x < \bar{x}\} = L^2(e^{x_L} - e^{\bar{x}})^2 P(t) + L^2 \int_{x_L}^{\bar{x}} (e^x - e^{\bar{x}})^2 u(x,t)dt =$$

$$(E(m_t) - (1-b))^2 P(t) + L^2\left\{f_2(x_L) - f_2(\bar{x}) - 2e^{\bar{x}}(f_1(x_L) - f_1(\bar{x})) + e^{2\bar{x}}(f_0(x_L) - f_0(\bar{x}))\right\}.$$

Incorporating the price of liquidity. It is easy to incorporate the price of liquidity in the above calculations. If the touching of the stop-loss level is accompanied by incurring the loss of α of the underlying asset position with the size $S_0 e^{x_L} = M_0(L - b)$, then

$$E(m_t) = Lf_1(x_L) - (L - 1) + (L - b)P(t) - \alpha(L - b)P(t) \quad ,$$

and

$$\sigma_<^2/2 = (E(m_t) - (1 - b - \alpha(L - b)))^2 P(t) +$$
$$L^2\left\{f_2(x_L) - f_2(\bar{x}) - 2e^{\bar{x}}(f_1(x_L) - f_1(\bar{x})) + e^{2\bar{x}}(f_0(x_L) - f_0(\bar{x}))\right\} \quad .$$

These formulae are used in figures in the main body of the article to illustrate the effects of stop-loss and price of liquidity.

References

[1] W.F. Sharpe (1964), "Capital asset prices: A theory of market equilibrium under conditions of risk", *Journal of Finance* **19 (3)**, 425-442;

J. Lintner (1965), "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics* **47 (1)**, 13-37;

J. Mossin (1966), "Equilibrium in a Capital Asset Market", *Econometrica* **34 (4)**, 768-783;

[2] S. Ross (1976), "The arbitrage theory of capital asset pricing", *Journal of Economic Theory* **13 (3)**, 341-360;

R. Roll, S. Ross (1980), "An empirical investigation of the arbitrage pricing theory", *Journal of Finance* **35**, 1073-1103;

[3] D. Kahneman, A. Tversky (1979), "Prospect Theory: An Analysis of Decision under Risk", *Econometrica* **XLVII**, 263-291;

A. Tversky, D. Kahneman (1992), "Advances in prospect theory: Cumulative representation of uncertainty", *Journal of Risk and Uncertainty* **5**, 297-323;

[4] H.A. Markowitz (1952), Portfolio Selection, *Journal of Finance* **6**, 7791;

[5] F. Black (1972), "Capital Market Equilibrium with Restricted Borrowing", *Journal of Business*, **45 (3)**, 444-455;

H. Levy, M. Sarnat (1984), Portfolio and Investment Selection: Theory and Practice, Prentice Hall, NY;

[6] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath (1999), "Coherent Measures of Risk", *Mathematical Finance* **9 (3)**, 203-228;

H. Follmer, A. Schied (2002), "Convex measures of risk and trading constraints", *Finance and Stochastics* **6 (4)**, 429-447;

[7] K. Ilinski (2001), Physics of Finance. Gauge modelling in non-equilibrium pricing, John Wiley & Sons.

[8] N.T. Chan, M. Getmansky, S.M. Haas, A.W. Lo (2005), "Systemic Risk and Hedge Funds", MIT Sloan Research Paper No. 4535-05 available at http://ssrn.com/abstract=671443;

[9] K. Ilinski (2009), "On optimal arbitrage trading", Fusion Asset Management LLP working paper;

[10] T. Adrian, H.S. Shin (2008), "Liquidity and Leverage", FRB of New York Staff Report No. 328;

[11] T. Adrian, H.S. Shin (2008), "Money, Liquidity and Monetary Policy", http://www.princeton.edu/ hsshin/www/AERpp29Dec2008.pdf;

[12] M.K. Brunnermeier (2009), "Deciphering the Liquidity and Credit Crunch 2007-2008", *Journal of Economic Perspectives* **23 (1)**, 77-100;

[13] P. McGuire, E. Remolona, K. Tsatsaronis (2005), "Time-varying exposures and leverage in hedge funds", BIS Quarterly Review, March 2005;

[14] P. McGuire, K. Tsatsaronis (2008), "Estimating hedge fund leverage", BIS Working Papers No 260, September 2008;

[15] V. Agarwal, N.D. Daniel, N.Y. Naik (2004), "Flows, Performance, and Managerial Incentives in Hedge Funds", EFA 2003 Annual Conference Paper No. 501, available at http://ssrn.com/abstract=424369;

[16] M. Schularick, A.M. Taylor (2009), "Credit Booms Gone Bust: Monetary Policy, Leverage Cycles and Financial Crises, 1870-2008", NBER Working Paper No. 15512, Issued in November 2009;

[17] R. Bertelli (2007), "Financial Leverage: Risks and Performance in Hedge Fund Strategies", available at SSRN: http://ssrn.com/abstract=1030400

[18] R.G. Rajan (2005), "Has Financial Development Made the World Riskier?", NBER Working Paper No. 11728, Issued in November 2005;

[19] T. Schneeweis, G.A. Martin, H.B. Kazemi, and V. Karavas (2005), "The Impact of Leverage on Hedge Fund Risk and Return", *The Journal of Alternative Investments* Spring 2005, **7 (4)**, 10-21;

[20] C.W. Gardiner (1997), Handbook of stochastic methods, Springer;

[21] Stochastic Processes. Formalism and applications. Lecture Notes in Physics, V.184. Springer, 1983.